SCHOOL OF MATHEMATICS AND PHYSICS

Lotka-Volterra Model

The Lotka-Voterra equations of interacting predator and prey systems, discussed in lectures, are unrealistic because they do not include the effect of limited resources on the food supply of the prey. Also, in the modern environment, prey are often culled or harvested. A more realistic system includes two extra terms:

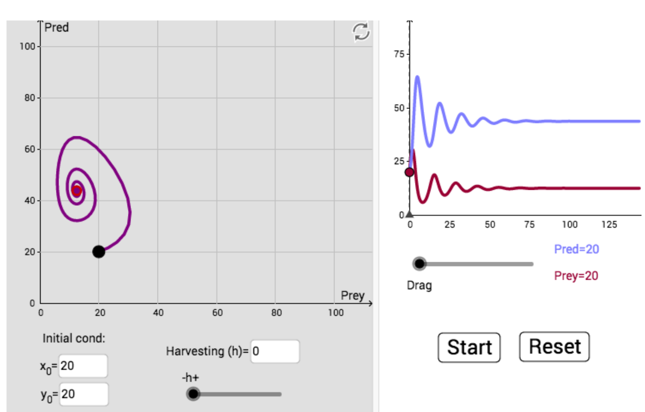
$$\begin{aligned} x' &= x(a - cx - dy) \\ y' &= -y(b - ex) - h \end{aligned}$$

where all of x, y, a, b, c, d, e, h are positive, and $a/c > b/e$. In this model we have that

- x represents the number of prey.
- y represents the number of predators.
- a is the growth rate of the prey.
- b is the death rate of the predators independent of the prey.
- d is the is the rate of consumption of the prey per predator.
- a/c is the carrying capacity of the prey independent of the predators.
- e is the growth rate of the predator per prey consumed.
- h is prey harvesting.

The following simulation demonstrates the solutions to these equations for specific values $a = 1,\ b = 0.25,\ c = 0.01,\ d = 0.02$ and $e = 0.02$. Click on the link bellow to access:

https://teaching.smp.uq.edu.au/scims/Appl_analysis/Lotka_Volterra.html



SCENARIO

In questions 1 and 2 imagine an actual situation where $a = 1,\ b = 0.25,\ c = 0.01,\ d = 0.02$ and $e = 0.02$ and $h = 0$ and there is danger prey may go extinct if the prey population levels dip below 4.

Use the simulation to help you answer the questions. Change the initial conditions within the simulation and drag the slider to analyse the values of prey and predator.

Question 1

Which of the following initial populations will probably mean that the prey go extinct?

(i) $x = 40$ prey and $y = 20$ predators.

(ii) $x = 30$ prey and $y = 20$ predators.

(iii) $x = 20$ prey and $y = 20$ predators.

(iv) $x = 60$ prey and $y = 5$ predators.

Question 2

A ranger decides to kill off half of the predators. Which of the following initial populations will probably mean that the prey go extinct?

(i) $x = 40$ prey and $y = 20$ to 10 predators.

(ii) $x = 30$ prey and $y = 20$ to 10 predators.

(iii) $x = 20$ prey and $y = 20$ to 10 predators.

(iv) $x = 60$ prey and $y = 5$ to 3 predators.

Question 3

Regular harvesting, say $h = 7$, can have unpredictable results. Which of the following initial conditions will show an unpredictable result? Which of them will probably mean that the prey go extinct?

(i) $x = 74$ prey and $y = 20$ predators.

(ii) $x = 30$ prey and $y = 25$ predators.

(iii) $x = 62$ prey and $y = 21$ predators.

(iv) $x = 20$ prey and $y = 8$ predators.

Question 4

Consider now the system without prey harvesting, that is

$$\begin{aligned} x' &= x(a - cx - dy) \\ y' &= -y(b - ex) \end{aligned} \tag{1}$$

where all of a, b, c, d, e are positive, and $a/c > b/e$.

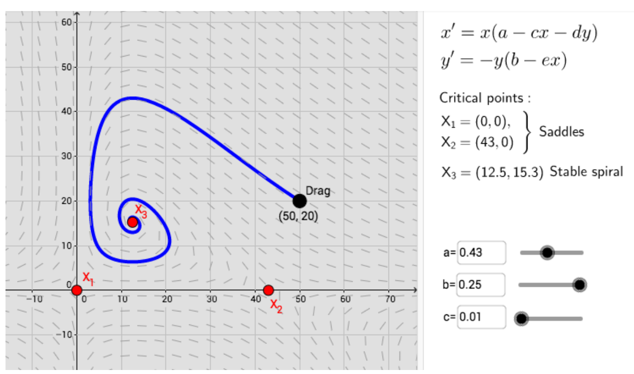
(a) Show that the system has critical points at

$$X_1 = (0,0), \quad X_2 = \left(\frac{a}{c}, 0\right), \quad \text{and} \quad X_3 = \left(\frac{b}{e}, \frac{ae - bc}{de}\right).$$

(b) Show that the critical points at X_1 and X_2 are saddle points, whereas the critical point at X_3 is either a stable node or a stable spiral point.

Use the simulation from the SciMS website to help you visualise and analyse the critical points. Click on the link bellow to access:

https://teaching.smp.uq.edu.au/scims/Appl_analysis/Lotka_Volterra_w.html



(c) [**Extra points**] Show that the point X_3 is a stable spiral point if

$$\frac{b}{e} < \frac{4ae}{c^2 + 4ce}.$$

Explain why this case will occur when the carrying capacity a/c of the prey is large.